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FUNDAMENTAL SOLUTIONS FOR STATIONARY VIBRATIONS OF AN ORTHOTROPIC ELASTIC MEDIUM

A. O. Vatul'yan and E. M. Chebakova

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An integral representation of the fundamental solution for stationary vibrations of an orthotropic medium is constructed, its asymptotic behavior is studied for large wavenumbers, and some special features of the wave-field structure are discussed.

Key words: *vibrations, orthotropic medium, fundamental solutions, asymptotic representation.*

Introduction. The boundary integral equation method is an effective tool for solving boundary-value elastic problems [1], which allows the dimension of the problem to be reduced by one. To formulate boundary integral equations, it is necessary to construct fundamental solutions [2]. In the isotropic theory of elasticity, the fundamental solutions for two-dimensional stationary vibrations are expressed in terms of the Hankel functions. In contrast, in the orthotropic case, explicit fundamental solutions cannot be constructed, but for practical applications, it suffices to obtain integral representations. Such a solution in the form of contour integrals in the complex plane was constructed in [3]. In the present paper, representations of the fundamental solutions for a two-dimensional orthotropic medium [4] are constructed in the form of integrals over a finite interval and their properties are studied.

Formulation of the Problem. We consider plane-strain stationary vibrations of an infinite orthotropic elastic medium induced by a point force applied to the point with coordinates (ξ_1, ξ_3). We denote the frequency of the vibrations by ω.

After elimination of the time factor $e^{-i\omega t}$, the equations of motion for the vibration amplitudes for an orthotropic material become

$$L_{ij}U_j^{(m)} + \rho\omega^2 U_i^{(m)} + \delta(x-\xi)\delta_{im} = 0 \qquad (m = 1,\ 3), \tag{1}$$

where ρ is the material density and L_{ij} are partial differential operators with constant coefficients:

$$L_{11} = C_{11}\,\partial_1^2 + C_{55}\,\partial_3^2, \qquad L_{33} = C_{55}\,\partial_1^2 + C_{33}\,\partial_3^2,$$

$$L_{13} = (C_{13} + C_{55})\,\partial_1\,\partial_3 = L_{31}, \qquad \partial_j = \partial/\partial x_j \quad (j = 1,\ 3)$$

(C_{ij} are the elastic constants of the orthotropic material subjected to the conditions of symmetry and positive-definiteness of the elastic energy). The problem of constructing fundamental solutions is closed by the radiation conditions at infinity formulated with the use of the limiting absorption principle [5].

Constructing the Solution. Using the operator method, system (1) can be reduced to the governing equations obtained in [6]. Integration of the latter is substantially simplified if the operator of these equations can be represented by superposition of two generalized metaharmonic operators of the second order and if a method of correction of their elastic constants can be used. It is worth noting that implementation of this method for constructing the fundamental solutions leads to a large error; therefore, below we use another method.

Using a two-dimensional Fourier integral transform, we obtain a solution of system (1) in the form [3, 4]

$$U_j^{(m)}(x_1, x_3, \xi_1, \xi_3) = \frac{1}{4\pi^2} \iint\limits_{\Gamma} \frac{p_{jm}(\alpha_1, \alpha_3, \omega)}{p_0(\alpha_1, \alpha_3, \omega)} \exp\left[i(\alpha_1(\xi_1 - x_1) + \alpha_3(\xi_3 - x_3))\right] d\alpha_1\, d\alpha_3. \tag{2}$$

Rostov State University, Rostov-on-Don 344104; a_lena_ch@mail.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 5, pp. 131–139, September–October, 2004. Original article submitted November 6, 2003.

Here $p_{jm}(\alpha_1, \alpha_3, \omega)$ and $p_0(\alpha_1, \alpha_3, \omega)$ are polynomials of the second and fourth orders, respectively:

$$p_{1m}(\alpha_1, \alpha_3, \omega) = \delta_{1m}(C_{55}\alpha_1^2 + C_{33}\alpha_3^2 - \rho\omega^2) - \delta_{3m}\alpha_1\alpha_3(C_{13} + C_{55}),$$

$$p_{3m}(\alpha_1, \alpha_3, \omega) = \delta_{3m}(C_{11}\alpha_1^2 + C_{55}\alpha_3^2 - \rho\omega^2) - \delta_{1m}\alpha_1\alpha_3(C_{13} + C_{55});$$

$$p_0(\alpha_1, \alpha_3, \omega) = (C_{11}\alpha_1^2 + C_{55}\alpha_3^2 - \rho\omega^2)(C_{55}\alpha_1^2 + C_{33}\alpha_3^2 - \rho\omega^2) - \alpha_1^2\alpha_3^2(C_{13} + C_{55})^2.$$

These polynominals possess the evenness property: $p_{jm}(-\alpha_1, -\alpha_3, \omega) = p_{jm}(\alpha_1, \alpha_3, \omega)$, $p_0(-\alpha_1, -\alpha_3, \omega) = p_0(\alpha_1, \alpha_3, \omega)$; Γ is a surface that coincides with the real plane R_2 except at the zeros of the polynomial $p_0(\alpha_1, \alpha_3, \omega)$, which are enveloped by the surface from below in accordance with the radiation conditions.

Curves of Polar Sets. Converting to the nondimensional variables

$$\alpha_j = k\beta_j, \quad k = \omega\Big(\frac{\rho}{C_{33}}\Big)^{1/2}, \quad \gamma_1 = \frac{C_{11}}{C_{33}}, \quad \gamma_5 = \frac{C_{55}}{C_{33}}, \quad \gamma_7 = \frac{C_{13}}{C_{33}}$$

and changing the variables $\beta_1 = \beta\cos\varphi$ and $\beta_3 = \beta\sin\varphi$, we write p_0 and p_{jm} as

$$p_0(\alpha_1, \alpha_3, \omega) = p_0(k\beta\cos\varphi, k\beta\sin\varphi) = C_{33}^2 k^4 p_0^*(\beta, \varphi),$$

$$p_{jm}(\alpha_1, \alpha_3, \omega) = p_{jm}(k\beta\cos\varphi, k\beta\sin\varphi) = C_{33}k^2 p_{jm}^*(\beta, \varphi) \qquad (j, m = 1, 3)$$

and analyze the zeros of the polynomial $p_0^*(\beta, \varphi)$. The polynomial is a biquadratic polynomial for the parameter β and can be written as

$$p_0^*(\beta, \varphi) = A(\varphi)(\beta^2 - \zeta_1^2(\varphi))(\beta^2 - \zeta_2^2(\varphi)),$$

where

$$\zeta_1^2(\varphi) = (b(\varphi) - \sqrt{D(\varphi)}\,)/(2A(\varphi)), \qquad \zeta_2^2(\varphi) = (b(\varphi) + \sqrt{D(\varphi)}\,)/(2A(\varphi)),$$

$$A(\varphi) = \gamma_1\gamma_5\cos^4\varphi + \cos^2\varphi\sin^2\varphi(\gamma_1 - 2\gamma_5\gamma_7 - \gamma_7^2) + \gamma_5\sin^4\varphi, \quad b(\varphi) = \gamma_5 + \sin^2\varphi + \gamma_1\cos^2\varphi,$$

$$D(\varphi) = [(\gamma_1 - \gamma_5)\cos^2\varphi + (\gamma_5 - 1)\sin^2\varphi]^2 + 4(\gamma_5 + \gamma_7)^2\sin^2\varphi\cos^2\varphi.$$

We consider the discriminant of the polynomial $p_0^*(\beta, \varphi)$ — the function $D(\varphi)$. One can easily infer that $D(\varphi) \geqslant 0$ and the discriminant vanishes only in two cases: 1) $\varphi = 0$ and $\gamma_1 = \gamma_5$; 2) $\varphi = \pi/2$ and $\gamma_5 = 1$. These cases are exceptional since the polynomial $p_0^*(\beta, \varphi)$ has multiple roots and will not be considered below. In the remaining cases, the discriminant is positive and the set of the zeros of the polynomial $p_0^*(\beta, \varphi)$ are two closed nonintersecting curves $\zeta_1(\varphi)$ and $\zeta_2(\varphi)$ that possess symmetry about both coordinate axes and intersect them at right angles. Both curves are fourth-order algebraic curves and, according to the general theory of fourth-order curves [7], the internal curve $\zeta_1(\varphi)$ is convex for any material and the number of possible inflection points of the external curve $\zeta_2(\varphi)$ should be equal to 8, 4 or 0. Budaev [8] gave a detailed classification of the curves according to the nondimensional parameters $\alpha = \gamma_5/\gamma_1$, $\beta = \gamma_5$, and $\gamma = 1 - (\gamma_7(\gamma_7 + 2\gamma_5))/\gamma_1$. He found that four different configurations of these curves are possible [in Fig. 1, curves 1 and 2 refer to the dependences $\zeta_1(\varphi)$ and $\zeta_2(\varphi)$, respectively]. It should be noted that cases are possible where the external-curve segments with negative curvature do not intersect the coordinate axes, intersect both axes or intersect one the axes. In the method of elastic-constant correction [6], the polar sets are always two ellipses, in contrast to the fourth-order curves considered above.

The nondimensional parameters γ_1, γ_5, and γ_7 define the curves $\zeta_1(\varphi)$ and $\zeta_2(\varphi)$ completely. We consider the configurations of these curves for various values of the parameter γ_5 and fixed parameters $\gamma_1 = 0.4$ and $\gamma_7 = 0.2$. Figures 2a and 2b show the curves $\zeta_1(\varphi)$ and $\zeta_2(\varphi)$, respectively, for small values $\gamma_5 = 0.1$–0.8 and Fig. 2c shows the curve $\zeta_1(\varphi)$ for large values $\gamma_5 = 5$–100. For small γ_5, the curves $\zeta_2(\varphi)$ have one common point with the polar coordinates $(0.615, 2.236)$ in the first quadrant, whereas the curves $\zeta_2(\varphi)$ almost coincide for large γ_5.

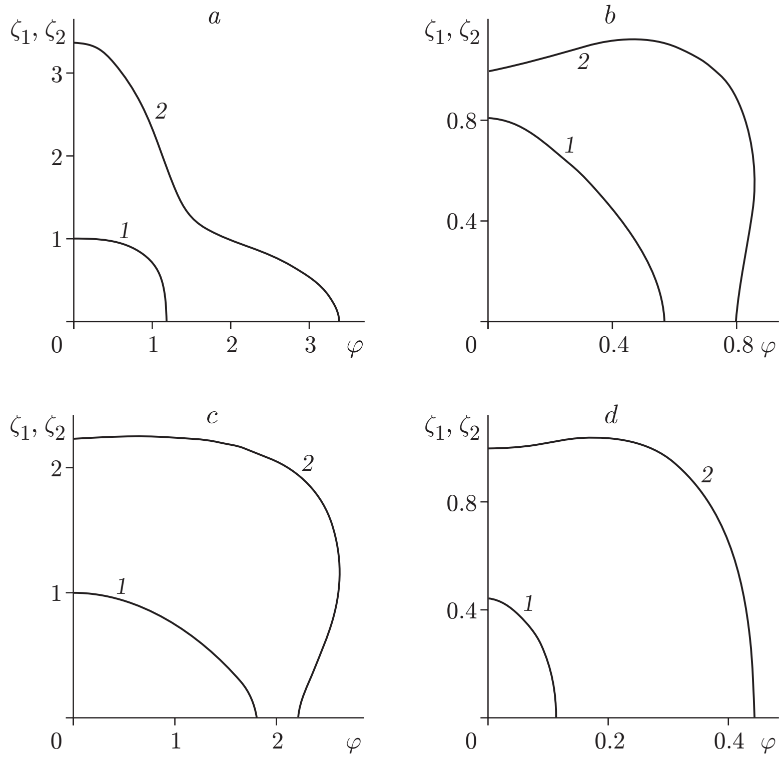

Fig. 1

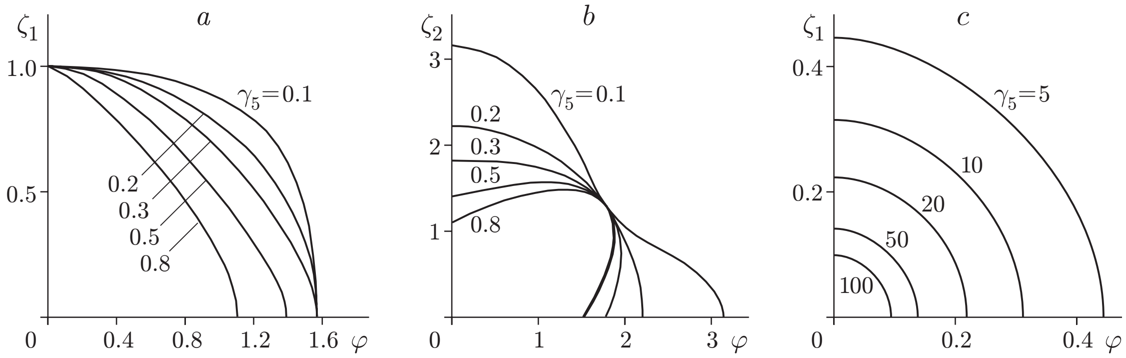

Fig. 2

Constructing Solutions in the Form of Single Integrals. To simplify (2), we write the integrand as

$$\frac{p^*_{jm}(\beta\cos\varphi,\beta\sin\varphi)}{p^*_0(\beta\cos\varphi,\beta\sin\varphi)}=\sum_{k=1}^{2}\frac{a_{kjm}(\varphi)}{A(\varphi)(\beta^2-\zeta_k^2(\varphi))},$$

where

$$a_{1jm}(\varphi)=\frac{B_{jm}-\zeta_1^2(\varphi)G_{jm}(\varphi)}{\zeta_2^2(\varphi)-\zeta_1^2(\varphi)},\qquad a_{2jm}(\varphi)=\frac{-B_{jm}+\zeta_2^2(\varphi)G_{jm}(\varphi)}{\zeta_2^2(\varphi)-\zeta_1^2(\varphi)},$$

$$G_{11}=\gamma_5\cos^2\varphi+\sin^2\varphi,\qquad G_{33}=\gamma_1\cos^2\varphi+\gamma_5\sin^2\varphi,$$

$$G_{13}=G_{31}=-(\gamma_5+\gamma_7)\sin\varphi\cos\varphi,\qquad B_{11}=B_{33}=1,\qquad B_{13}=B_{31}=0.$$

We note that for any orthotropic materials, the functions possess the following properties:

$$a_{kjm}(\varphi+\pi)=a_{kjm}(\varphi),\qquad a_{kjj}(-\varphi)=a_{kjj}(\varphi),\qquad a_{k13}(-\varphi)=-a_{k13}(\varphi),$$

$$\zeta_k(\pi+\varphi)=\zeta_k(\varphi)=\zeta_k(-\varphi),\qquad k=1,2,\quad j,m=1,3.$$

We transform (2) by converting to the polar coordinates

$$|x-\xi|=r,\qquad \xi_1-x_1=r\cos\psi,\qquad \xi_3-x_3=r\sin\psi.$$

By virtue of elastic symmetry, we assume that $\psi\in[0,\pi/2]$. As a result, we obtain

$$U_j^{(m)}(r,\psi)=\frac{1}{(2\pi)^2C_{33}}\int\limits_{\sigma_+}\int\limits_0^{2\pi}\sum_{k=1}^{2}\frac{a_{kjm}(\varphi)}{A(\varphi)(\beta^2-\zeta_k^2(\varphi))}\exp\left(ik\beta r\cos\left(\varphi-\psi\right)\right)\beta\,d\beta\,d\varphi$$

$$=\frac{1}{(2\pi)^2C_{33}}\int\limits_{\sigma_+}\int\limits_0^{\pi}\sum_{k=1}^{2}\frac{a_{kjm}(\varphi)}{A(\varphi)(\beta^2-\zeta_k^2(\varphi))}$$

$$\times\Big[\exp\left(ik\beta r\cos\left(\varphi-\psi\right)\right)+\exp\left(-ik\beta r\cos\left(\varphi-\psi\right)\right)\Big]\beta\,d\beta\,d\varphi$$

(σ_+ is the part of the cross section of the surface Γ formed by the plane $\varphi=\text{const}$ that lies in the right half-plane and envelops the poles of the integrand from below). The integrals over the contour σ_+ are evaluated by the method outlined in [9].

We consider the integral

$$I_2(z,\zeta)=\int\limits_{\sigma_+}\frac{\exp\left(i\beta z\right)+\exp\left(-i\beta z\right)}{\beta^2-\zeta^2}\,\beta\,d\beta.$$

To evaluate it, we introduce the contours

$$C_\Gamma^+=\sigma_R^+\cup C_R^+\cup[iR,0],\qquad C_\Gamma^-=\sigma_R^+\cup C_R^-\cup[-iR,0]$$

(C_R^+ and C_R^- are parts of the circumference of radius R with center at the coordinate origin that lie in the first and fourth quadrants, respectively, and σ_R^+ is a segment of the contour σ_+ located inside the circumference of radius R).

Using contour integration and Jordan's lemma [10], we obtain

$$I_2(z,\zeta)=\pi i\exp\left(iz\zeta\right)+\int\limits_0^{\infty}\frac{\exp\left(-tz\zeta\right)}{t^2+1}\,t\,dt=\pi i\exp\left(iz\zeta\right)-\left[\operatorname{ci}\left(z\zeta\right)\cos\left(z\zeta\right)+\operatorname{si}\left(z\zeta\right)\sin\left(z\zeta\right)\right],$$

where $\operatorname{ci}(x)$ and $\operatorname{si}(x)$ are integral cosine and sine, respectively [11]:

$$\operatorname{ci}(\gamma)=C+\ln\gamma+\int\limits_0^{\gamma}\frac{\cos t-1}{t}\,dt,\qquad \operatorname{si}(\gamma)=-\frac{\pi}{2}+\int\limits_0^{\gamma}\frac{\sin t}{t}\,dt$$

(C is the Euler constant).

In a similar manner, we calculate $I_2(z,\zeta)$ for $z<0$. Combining these two cases, we arrive at the following representation of the fundamental solutions:

$$U_j^{(m)}(r,\psi)=\frac{1}{2\pi^2C_{33}}\int\limits_0^{\pi}\sum_{k=1}^{2}\frac{a_{kjm}(\varphi)}{A(\varphi)}\,F_2(t_k(r,\varphi,\psi))\,d\varphi,\tag{3}$$

$$F_2(z)=(\pi i/2)\,e^{i|z|}-\operatorname{ci}|z|\cos|z|-\operatorname{si}|z|\,\sin|z|,\qquad t_k(r,\varphi,\psi)=kr\zeta_k(\varphi)\cos\left(\varphi-\psi\right).$$

For an isotropic material, we have

$$\gamma_1=1,\qquad \gamma_5=(1-2\nu)/(2(1-\nu)),\qquad \gamma_7=\nu/(1-\nu)$$

(ν is Poisson's ratio), $\zeta_1(\varphi)$ and $\zeta_2(\varphi)$ are circumferences, and representation (3) becomes the well-known representation in terms of Hankel functions [1].

We note that the fundamental solutions constructed above possess the following symmetry properties:

$$U_j^{(m)}(r,\psi) = U_j^{(m)}(r,\pi+\psi) \quad (j,\ m=1,\ 3), \qquad U_j^{(m)}(r,-\psi) = U_j^{(m)}(r,\psi) \quad (j=m),$$

$$U_j^{(m)}(r,-\psi) = -U_j^{(m)}(r,\psi) \qquad (j \neq m).$$

Constructing Asymptotic Representations. We study the structure of the fundamental solutions (3) for small and large values of r using an asymptotic expansions of special functions.

Bearing in mind that $\mathrm{si}\,(z) \sim -\pi/2$ and $\mathrm{ci}\,(z) \sim C + \ln z$ as $z \to 0$ [11], we obtain the following asymptotic representation of the fundamental solutions (3) for $r \to 0$:

$$U_j^{(m)}(r,\psi) = -\frac{\ln(kr)}{2\pi^2 C_{33}} \int_0^{\pi} \sum_{k=1}^{2} \frac{a_{kjm}(\varphi)}{A(\varphi)}\,d\varphi + O(1), \qquad j,m=1,3. \tag{4}$$

It follows from (4) that the leading term of the asymptotic representation of the fundamental solutions has a logarithmic singularity for $r \to 0$, as in the isotropic case.

Taking into account the estimate $-\,\mathrm{ci}\,|z|\cos|z| - \mathrm{si}\,|z|\sin|z| = O(1/z^2)$ for $|z| \to \infty$ [11] and using the properties of the functions $a_{kjm}(\varphi)$, $A(\varphi)$, and $\zeta_k(\varphi)$, one can show that the following two integrals make the main contribution to the asymptotic representation of the fundamental solutions for large r:

$$U_j^{(m)}(r,\psi) = \frac{i}{4\pi C_{33}} \Big(\int_0^{\psi+\pi/2} \sum_{k=1}^{2} \frac{a_{kjm}(\varphi)}{A(\varphi)} \exp\left[ikr\zeta_k(\varphi)\cos(\varphi-\psi)\right] d\varphi$$

$$+ \int_{\psi+\pi/2}^{\pi} \sum_{k=1}^{2} \frac{a_{kjm}(\varphi)}{A(\varphi)} \exp\left[-ikr\zeta_k(\varphi)\cos(\varphi-\psi)\right] d\varphi \Big), \qquad j,m=1,3. \tag{5}$$

We construct the asymptotic representation for $r \to +\infty$ using the stationary phase method [12]. To find the stationary points of the phase functions

$$S_k^{+}(\varphi,\psi) = \zeta_k(\varphi)\cos(\varphi-\psi), \qquad \varphi \in [0, \psi+\pi/2],$$

$$S_k^{-}(\varphi,\psi) = -\zeta_k(\varphi)\cos(\varphi-\psi), \qquad \varphi \in [\psi+\pi/2, \pi] \qquad (k=1,2)$$

for various values of the polar angle $\psi \in [0,\pi/2]$, we employ a numerical method to solve the equations

$$\zeta_k'(\varphi)\cos(\varphi-\psi) - \zeta_k(\varphi)\sin(\varphi-\psi) = 0, \qquad k=1,2.$$

Figures 3a and 3b show the values of the stationary points $\varphi_s(\psi)$ of the phase functions $S_k^{\pm}(\varphi,\psi)$ for $k=1$ and 2, respectively. For calculations, the following nondimensional parameters of the material ($BaSO_4$) were used: $\gamma_1 = 0.687$, $\gamma_5 = 0.0865$, and $\gamma_7 = 0.313$.

One can see from Fig. 3 that for each value of the polar angle $\psi \in [0,\pi/2]$, one value of the stationary point of the phase functions $S_1^{\pm}(\varphi,\psi)$ always exists. There are several ranges of the polar angle, in each of which the stationary point of the phase functions $S_2^{\pm}(\varphi,\psi)$ has one value or three values. Namely, one value was found for $\psi \in [0,\psi_1) \cup (\psi_2,\pi/2]$ and three values were found for $\psi \in (\psi_1,\psi_2)$; on each boundary of these regions ($\psi=\psi_1$ and $\psi=\psi_2$), two values of the stationary points exist. One of these is multiple and satisfies the system

$$S_{\varphi}^{\pm\prime}(\varphi,\psi) = 0, \quad S_{\varphi\varphi}^{\pm\prime\prime}(\varphi,\psi) = 0, \quad S_{\varphi\varphi\varphi}^{\pm\prime\prime\prime}(\varphi,\psi) \neq 0, \qquad \varphi \in [0,\pi],\ \psi \in [0,\pi/2],$$

which corresponds to a degenerate value of the stationary point. The ψ direction corresponding to the solution of this system will be called the critical direction. We note that the curvature $æ(\varphi)$ of the function $\zeta_2(\varphi)$ vanishes for this value of ψ.

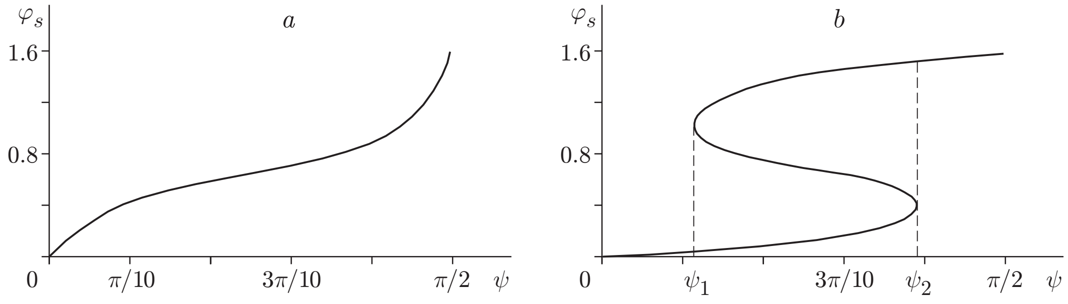

Fig. 3

Applying the stationary phase method to each integral in (5), we obtain the following formula for the main part of the asymptotic representation $U_j^{(m)}(r,\psi)$ for $r \to +\infty$:

$$U_j^{(m)}(r,\psi) = \frac{i}{4\pi C_{33}}\Big(\sqrt{\frac{2\pi}{kr}}\sum_{k=1}^{2}\sum_{s=1}^{M}\sqrt{\frac{1}{|S_k^{+\prime\prime}(\varphi_s(\psi),\psi)|}}\Big[\frac{a_{kjm}(\varphi_s(\psi))}{A(\varphi_s(\psi))} + O((kr)^{-1})\Big]$$
$$\times \exp\Big[ikrS_k^{+}(\varphi_s(\psi),\psi) + \frac{i\pi}{4}\operatorname{sign}\,(S_k^{+\prime\prime}(\varphi_s(\psi),\psi))\Big]$$
$$+\sqrt{\frac{2\pi}{kr}}\sum_{k=1}^{2}\sum_{s=1}^{M}\sqrt{\frac{1}{|S_k^{-\prime\prime}(\varphi_s(\psi),\psi)|}}\Big[\frac{a_{kjm}(\varphi_s(\psi))}{A(\varphi_s(\psi))} + O((kr)^{-1})\Big]$$
$$\times \exp\Big[ikrS_k^{-}(\varphi_s(\psi),\psi) + \frac{i\pi}{4}\operatorname{sign}\,(S_k^{-\prime\prime}(\varphi_s(\psi),\psi))\Big]\Big), \quad j,m = 1,3. \tag{6}$$

In (6), the upper limit of the summation M can be equal to 1, 2, or 3, depending on the number of values of the stationary points. For each fixed value of $\psi \in [0,\pi/2]$, the phase functions $S_k^{+}(\varphi,\psi)$ and $S_k^{-}(\varphi,\psi)$ have stationary points that lie within the intervals $[0,\psi+\pi/2]$ and $[\psi+\pi/2,\pi]$, respectively.

The contribution to the main part of the asymptotic representation (6) to the critical directions is given by

$$U_j^{(m)}(r,\psi_{1,2}) = \frac{i}{4\pi C_{33}}\,\frac{\Gamma(1/3)}{3}\,(kr)^{-1/3}\Big[\frac{a_{2jm}(\varphi_s(\psi_{1,2}))}{A(\varphi_s(\psi_{1,2}))} + O((kr)^{-1/3})\Big]$$
$$\times \exp\Big[ikrS_2^{\pm}(\varphi_s(\psi_{1,2}),\psi_{1,2}) + \frac{i\pi}{6}\operatorname{sign}\,S_2^{\pm\prime\prime\prime}(\varphi_s(\psi_{1,2}),\psi_{1,2})\Big], \qquad j,m = 1,3.$$

The main part of the asymptotic representation of the fundamental solutions $U_j^{(m)}(r,\psi)$ is written as

$$\begin{aligned} U_j^{(m)}(r,\psi) &= \frac{1}{\sqrt{kr}}\sum_{k=1}^{2}A_{jmk}(\psi)\exp\,[ikrS_k(\psi)], \qquad \psi \in [0,\psi_1)\cup(\psi_2,\pi/2],\\ U_j^{(m)}(r,\psi) &= \frac{1}{\sqrt{kr}}\sum_{k=1}^{4}A_{jmk}(\psi)\exp\,[ikrS_k(\psi)], \qquad \psi \in (\psi_1,\psi_2) \qquad (j,m=1,3), \end{aligned} \tag{7}$$

where

$$A_{jm1}(\psi) = \sqrt{\frac{2}{\pi}}\,\frac{i}{4C_{33}}\sqrt{\frac{1}{|S_1''(\varphi_*(\psi),\psi)|}}\,\frac{a_{1jm}(\varphi_*(\psi))}{A(\varphi_*(\psi))}\,\exp\Big[\frac{i\pi}{4}\operatorname{sign}\,S_1(\varphi_*(\psi),\psi)\Big],$$

$$A_{jmk}(\psi) = \sqrt{\frac{2}{\pi}}\,\frac{i}{4C_{33}}\sqrt{\frac{1}{|S_k''(\varphi_{k-1}(\psi),\psi)|}}\,\frac{a_{kjm}(\varphi_{k-1}(\psi))}{A(\varphi_{k-1}(\psi))}\,\exp\Big[\frac{i\pi}{4}\operatorname{sign}\,S_k(\varphi_{k-1}(\psi),\psi)\Big] \qquad (k=2,3,4)$$

are the amplitudes, and

$$S_1(\psi) = \zeta_1(\varphi_*(\psi))\cos\,(\varphi_*(\psi)-\psi), \quad S_k(\psi) = \zeta_2(\varphi_{k-1}(\psi))\cos\,(\varphi_{k-1}(\psi)-\psi) \qquad (k=2,3,4)$$

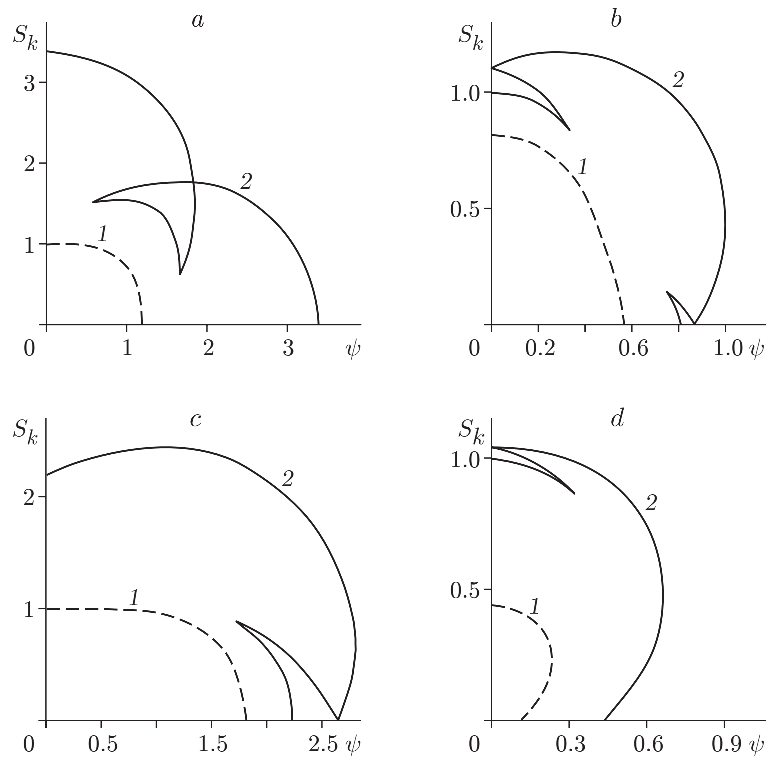

Fig. 4

are the vibration phases. Figure 4 shows curves of the phases versus the polar angle in the first quadrant (by virtue of the symmetry of the phase functions about the coordinate axes). These dependences correspond to four different configurations of the curves $\zeta_k(\varphi)$ ($k = 1, 2$) given in Fig. 1. In Fig. 4a, the functions $S_1(\psi)$ and $S_2(\psi)$ are presented in the regions $\psi \in [0, \psi_1) \cup (\psi_2, \pi/2]$ (curves 1 and 2, respectively) and the functions $S_k(\psi)$ in the region $\psi \in (\psi_1, \psi_2)$ (curve 1 refers to $k = 1$ and curve 2 refer to $k = 2$, 3, and 4). On the segment $\psi \in (\psi_1, \psi_2)$, the functions $S_3(\psi)$ and $S_4(\psi)$ form a triangular lobe at the center of the figure. In Fig. 4b–d, the lobes formed by $S_3(\psi)$ and $S_4(\psi)$ intersect the coordinate axes and the halves of the corresponding curvilinear triangles are depicted on them.

From (7) it follows that there are different numbers of propagating vibration modes (from one to four) in different ranges of the angle ψ. We note that it is impossible to obtain this structure of the wave field in a far zone by the method of elastic-constant correction [6] since it gives two traveling waves for all directions.

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